ON TOTAL INCOMPARABILITY OF

MIXED TSIRELSON SPACES

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ABSTRACT

We give criteria of total incomparability for certain classes of mixed Tsirelson spaces. We show that spaces of the form $T[(\mathcal{M}_k,\theta_k)_{k=1}^{\ell}]$ with index $i(\mathcal{M}_k)$ finite are either c_0 or ℓ_p saturated for some p and we characterize when any two spaces of such a form are totally incomparable in terms of the index $i(\mathcal{M}_k)$ and the parameter θ_k. Also, we give sufficient conditions of total incomparability for a particular class of spaces of the form $T[(\mathcal{A}_k,\theta_k)_{k=1}^{\infty}]$ in terms of the asymptotic behaviour of the sequence $\|\sum_{i=1}^{n}e_i\|$ where (e_i) is the canonical basis.

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0. Introduction

Denote by c_{00} the vector space of all real valued sequences which are eventually zero and by $(e_i)_{i=1}^{\infty}$ its usual unit vector basis. For $E\subset\mathbb{N}$ and $x=\sum_{i=1}^{\infty}a_ie_i\in c_{00}$ we denote $Ex=\sum_{i\in E}a_ie_i$. Also, for finite subsets $E,F\subseteq\mathbb{N}$, we write $E<F$ (or $E\le F$) if $\max E<\min E$ ($\max E\le\min E$). For simplicity, we write $n\le E$ instead of $\{n\}\le E$.

Mixed Tsirelson spaces were introduced in full generality in [2]. We can define those spaces, denoted by $T[(\mathcal{M}_k,\theta_k)_{k\in I}]$, as the completion of c_{00} under a norm which satisfies an implicit equation of the following kind:

$$\|x\|=\max\left\{\|x\|_{\infty},\sup_{k\in I}\left\{\theta_k\sup_{n\in\mathbb{N}}\left\{\sum_{i=1}^{n}\|E_ix\|\mid (E_i)_{i=1}^{n}\ \mathcal{M}_k-admissible\right\}\right\}\right\},\qquad x\in c_{00}$$

where the $\mathcal{M}_k$'s are certain (see Definition 4 below) families of finite subsets of $\mathbb{N}$, $\theta_k\in(0,1]$ for all $k\in I\subseteq\mathbb{N}$ and $(E_i)_{i=1}^{n}$ is $\mathcal{M}_k$-admissible if there exists $\{m_1,\dots,m_n\}\in\mathcal{M}_k$ such that $m_1\le E_1<m_2\le E_2<\dots<m_n\le E_n$.

The first remarkable space in this class is the so called Tsirelson space, introduced by Figiel and Johnson [7] in 1974. (It is actually the dual of the space originally constructed by Tsirelson in [12].) In our notation this space is $T[\mathcal{S},1/2]$, where $\mathcal{S}$ is Schreier's class, that is, the set of subsets of $\mathbb{N}$ of cardinality smaller than their first element. Since its construction it was usually considered a "pathological space", a place to look for counterexamples to statements in the Banach space theory. In fact, the reason why it was constructed was to provide a counterexample to the assertion "every Banach space contains c_0 or ℓ_p for some $1\le p<\infty$".

The second space of the class is Tzafriri space, introduced in 1979 in [13] ($T[(\mathcal{A}_k,\gamma/\sqrt{k})_{k\in\mathbb{N}}],0<\gamma<1$ in our notation where $\mathcal{A}_k$ is the set of subsets of $\mathbb{N}$ of at most k elements), also constructed as a counterexample to a statement in the Banach space theory. In 1991 a third example, namely the

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Schlumprecht space $T[(\mathcal{A}_k, 1/\log_2(1+k))_{k\in\mathbb{N}}]$, was considered, see [11], and with its help a fruitful period started when many "classical" problems in the infinite dimensional Banach space theory were solved, such as the distortion problem or the unconditional basic sequence problem.

A common feature of the three Banach spaces mentioned above is that they do not contain any $\ell_p, 1 \le p < \infty$ or c_0. (Actually, in the case of Tzafriri spaces this has been proved, as far as we know, only for $0 < \gamma < 10^{-6}$, see [6].) Moreover, since $\ell_p, 1 \le p < \infty$ and c_0 are minimal (recall that a Banach space X is minimal if every subspace of X contains a further subspace isomorphic to X) it easily follows that they are totally incomparable to any of the three examples above (recall that two Banach spaces X and Y are totally incomparable if no subspace of X is isomorphic to any of Y). We use the word "subspace" here and throughout the paper for "closed infinite dimensional subspace".

In 1986 Bellenot [3] showed that $\ell_p, 1 \le p < \infty$ and c_0 are isomorphic to mixed Tsirelson spaces of the form $T[(\mathcal{A}_n, \theta)], \theta \in (0, 1]$. This was somewhat surprising as it showed that $\ell_p, 1 \le p < \infty$ and c_0 belong to a class of spaces up to then considered pathological.

It is well known that $\ell_p, 1 \le p < \infty$ and c_0 are totally incomparable to each other. Moreover, ℓ_p and c_0 and the three examples, with $0 < \gamma < 10^{-6}$ in the case of Tzafriri space, are all totally incomparable to each other (see [6] for the details and also use the minimality of the Schlumprecht space). This shows that, at least in the examples considered, the modification of the $\theta_k' s$ or the $\mathcal{M}_k' s$ produce totally incomparable spaces.

In the first section we discuss in full generality the case when $\theta_k = 1$ for some k. In this case, the spaces c_0 and ℓ_1 will play a crucial role.

In the second section we consider mixed Tsirelson spaces of the form $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}], \theta_k \in (0, 1)$, with index $i(\mathcal{M}_k)$, as defined in [2], finite and we characterize when any two spaces of such a form are totally incomparable. This is done by following the ideas in [4] and showing that every such space is either c_0 or ℓ_p saturated for some p. Recall that given a Banach space Y, a Banach space X is Y saturated if every subspace of X contains a further subspace isomorphic to Y.

In the third section we focus on spaces of the form $T[(\mathcal{A}_k, \theta_k)_{k=1}^{\infty}], \theta_k \in (0, 1]$, such that ℓ_1 is finitely block represented in every block subspace. We give sufficient conditions of total incomparability in terms of the asymptotic behaviour of the sequence $\| \sum_{i=1}^{n} e_i \|$ where (e_i) is the canonical basis. These conditions apply to cases different from those considered in [9].

Notation. If K is a subset of a Banach space X, $\overline{Span}\{K\}$ denotes the closure of the algebraic linear span of K. If $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$, the support of x is the set $\mathrm{supp}(x) = \{i \in \mathbb{N} \mid a_i \neq 0\}$. For $x, y \in c_{00}$ we write $x < y$ if $\mathrm{supp}(x) < \mathrm{supp}(y)$. We say that $E_1, \ldots, E_n \subset \mathbb{N}$ are successive if $E_1 < E_2 < \ldots < E_n$. The vectors $x_1, \ldots, x_n$ are successive if their supports are. A block sequence (x_i) is a sequence of successive vectors. The cardinality of a set E is denoted by $|E|$. The standard norm of $\ell_p, 1 \le p \le \infty$ is denoted by $\|\cdot\|_p$. Other unexplained notation is standard and can be found for instance in [8].

Definition 1. *Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$. We say that $\mathcal{M}$ is compact if the set $\{\aleph_E \mid E \in \mathcal{M}\}$ is a compact subset of the Cantor set $\{0,1\}^{\mathbb{N}}$ with the product topology.*

Remark 1. *In Definition 1, $\{0,1\}^{\mathbb{N}}$ is identified with the space of all mappings $f : \mathbb{N} \longrightarrow \{0,1\}$ and $\aleph_E$ is the characteristic function of E. In $\{0,1\}^{\mathbb{N}}$, the convergence under the product topology is the pointwise convergence. Therefore if $E \subseteq \mathbb{N}$ is a finite set and $\aleph_{E_k}$ converges to $\aleph_E$ pointwise, there exists $N \in \mathbb{N}$ such that $E \subseteq E_k$ for all $k \ge N$.*

Definition 2. *Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$. We say that $\mathcal{M}$ is hereditary if $E \in \mathcal{M}$ and $F \subseteq E$ implies that $F \in \mathcal{M}$.*

Definition 3. *Let $\mathcal{M}$ be a compact family of finite subsets of $\mathbb{N}$. We define a transfinite sequence $(\mathcal{M}^{(\lambda)})$ of subsets of $\mathcal{M}$ as follows:*

1. *$\mathcal{M}^{(0)} = \mathcal{M}$.*
2. *$\mathcal{M}^{(\lambda+1)} = \{E \in \mathcal{M} \mid \aleph_E$ is a limit point of the set $\{\aleph_E | E \in \mathcal{M}^{(\lambda)}\}\}$.*
3. *If λ is a limit ordinal then $\mathcal{M}^{(\lambda)} = \bigcap_{\mu<\lambda} \mathcal{M}^{(\mu)}$.*

We call the least λ for which $\mathcal{M}^{(\lambda)} \subseteq \{\emptyset\}$ the index of $\mathcal{M}$ and denote it by $i(\mathcal{M})$.

Definition 4. *Let $I \subseteq \mathbb{N}$. Let $(\mathcal{M}_k)_{k\in I}$ be a sequence of compact hereditary families of finite subsets of $\mathbb{N}$ and let $(\theta_k)_{k\in I} \subset (0,1]$. We denote by $T\left[(\mathcal{M}_k,\theta_k)_{k\in I}\right]$ the completion of c_{00} with respect to the norm defined by*

$$\|x\| = \max\left\{\|x\|_\infty, \sup_{k\in I}\left\{\theta_k \sup_{n\in\mathbb{N}}\left\{\sum_{i=1}^n \|E_i x\| \mid (E_i)_{i=1}^n\ \mathcal{M}_k - admissible\right\}\right\}\right\}$$

and we call it the mixed Tsirelson space defined by the sequence $(\mathcal{M}_k,\theta_k)_{k\in I}$.

Remark 2. *The existence of such a norm is shown, for instance, in [10]. It follows from the definition of the norm that the sequence $(e_i)_{i=1}^\infty$ is a normalized 1-unconditional basis for $T\left[(\mathcal{M}_k,\theta_k)_{k\in I}\right]$.*

Remark 3. *There are two useful alternative ways to define the norm. Given $x = \sum_{n=1}^\infty a_n e_n \in c_{00}$,*

(i) define a non decreasing sequence of norms on c_{00}:

$$|x|_0 = \max_{n\in\mathbb{N}} |a_n|$$
$$|x|_{s+1} = \max\left\{|x|_s, \sup_{k\in I}\left\{\theta_k \sup_{n\in\mathbb{N}}\left\{\sum_{i=1}^n |E_i x|_s \mid (E_i)_{i=1}^n\ \mathcal{M}_k - admissible\right\}\right\}\right\}$$

Then $\|x\| = \sup_{s\in\mathbb{N}\cup\{0\}} |x|_s$.

(ii) Let $K_0 = \{\pm e_n \mid n\in\mathbb{N}\}$. Given K_s, $s\in\mathbb{N}\cup\{0\}$, let

$$K_{s+1} = K_s \cup \Big\{\theta_k\cdot(f_1+\ldots+f_d) \mid k\in I, d\in\mathbb{N}, f_i\in K_s, i=1,\ldots,d$$
$$\text{are successive and } (\mathrm{supp}(f_1),\ldots,\mathrm{supp}(f_d))\ \mathcal{M}_k - admissible\Big\}$$

Let $K = \bigcup_{s=0}^\infty K_s$. Then $\|x\| = \sup\{f(x) \mid f\in K\}$.

The latter definition of the norm provides information about the dual space. Looking at the set K as a set of functionals it is not difficult to see that B_{X^*} is the closed convex hull of K, where the closure is taken either in the weak-$*$ topology or in the pointwise convergence topology.

1. The case $\theta_k = 1$

Let $J = \{k\in I \mid \theta_k = 1\}$. If J is not empty, we give information about the structure of $T\left[(\mathcal{M}_k,\theta_k)_{k\in I}\right]$ depending on the index $i(\mathcal{M}_k), k\in J$. It is known that if $i(\mathcal{M}_k)\geq 2$ for some $k\in J$, then $T\left[(\mathcal{M}_k,\theta_k)_{k\in I}\right]$ contains an isomorphic copy of ℓ_1. Actually it is possible to say much more as our next proposition shows.

Proposition 1. *If $i(\mathcal{M}_{k_0})\geq 2$ for some $k_0\in J$, then $T\left[(\mathcal{M}_k,\theta_k)_{k\in I}\right]$ is ℓ_1-saturated.*

Proof: By the Bessaga-Pelczynski principle (see e.g. [6], pg. 10), it suffices to show that every block subspace contains a further subspace isomorphic to ℓ_1. Recall that a block subspace is a space of the form $\overline{Span}\{u_i, i\in\mathbb{N}\}$, with $(u_i)_{i=1}^\infty$ a block sequence.

Let $(u_i)_{i=1}^\infty$ be a block sequence. We are going to construct a subsequence $(u_{i_k})_{k=1}^\infty$ of $(u_i)_{i=1}^\infty$ equivalent to the ℓ_1 basis.

Let $\{p\}\in\mathcal{M}_{k_0}^{(1)}$. We can choose u_{i_1} such that $p<u_{i_1}$. Now, since $\{p\}\in\mathcal{M}_{k_0}^{(1)}$, there exists $n_1\in\mathbb{N}$ such that $n_1>u_{i_1}$ and $\{p,n_1\}\in\mathcal{M}_{k_0}$, so we can take u_{i_2} such that $n_1<u_{i_2}$. Continuing in this manner, we can construct a subsequence $(u_{i_k})_{k=1}^\infty$ of $(u_i)_{i=1}^\infty$ such that for every $k\in\mathbb{N}$ there exists $n_k\in\mathbb{N}$ such that $u_{i_k}<n_k<u_{i_{k+1}}$ and $\{p,n_k\}\in\mathcal{M}_{k_0}$. It is now easy to see that $(u_{i_k})_{k=1}^\infty$ is equivalent to the ℓ_1 basis. ∎

The following example shows a Tsirelson type space ℓ_1-saturated but not isomorphic to ℓ_1. It was shown to us by I. Deliyanni.

Example 1. *Let $\mathcal{M} = \{F \subseteq \mathbb{N} \mid \exists i \in \mathbb{N}$ such that $F \subseteq \{1, 2^i\}\}$ and $\theta = 1$.*

It is clear that $i(\mathcal{M}) = 2$. If $T[\mathcal{M}, \theta]$ were isomorphic to ℓ_1 then since ℓ_1 has a unique − up to equivalence − normalized unconditional basis, there would exist a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\frac{1}{C}\sum_{i=1}^{n} |a_i| \le \left\| \sum_{i=1}^{n} a_i e_i \right\| \le C \sum_{i=1}^{n} |a_i| .$$

Now taking $x = \displaystyle\sum_{i=2^k+1}^{2^{k+1}} e_i$ we would obtain $2^k - 1 \le C$ for all $k \in \mathbb{N}$.

We now examine $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ with $i(\mathcal{M}_k) = 1, k \in J$. We will find different subspaces depending on whether the set $\displaystyle\bigcup_{k\in J} \mathcal{M}_k$ contains only a finite number of non singleton sets or not.

Proposition 2. *Let $I' \subseteq I$ be such that $\displaystyle\bigcup_{k\in I'} \mathcal{M}_k$ contains only a finite number of non singleton sets.*

(1) If $I' \neq I$, then $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ is isomorphic to $T\left[(\mathcal{M}_k, \theta_k)_{k\in I\setminus I'}\right]$.
(2) If $I' = I$, then $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ is isomorphic to c_0.

Proof: (1). Let $\|\cdot\|$ and $\|\cdot\|'$ be the norms of the spaces $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ and $T\left[(\mathcal{M}_k, \theta_k)_{k\in I\setminus I'}\right]$, respectively. We will see that they are equivalent. Clearly, $\|\cdot\|' \le \|\cdot\|$.

For the other inequality let $M = \max\left\{\max E \mid E \in \displaystyle\bigcup_{k\in I'} \mathcal{M}_k, \text{ non singleton}\right\}$ and write $x = \displaystyle\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{M} a_i e_i + \sum_{i=M+1}^{\infty} a_i e_i := x_1 + x_2$.

We have $\|x_1\| \le M \|x\|'$ since $\|x_1\| = \left\|\displaystyle\sum_{i=1}^{M} a_i e_i\right\| \le \sum_{i=1}^{M} |a_i| \le \sum_{i=1}^{M} \|x\|_\infty \le M \|x\|'$.

On the other hand, we show first by induction over s that $|x_2|_s \le |x_2|'_s$. For $s = 0$ it is clear. Suppose now that it is true for s and let $E_1, \dots, E_n$ be a sequence of finite subsets of $\mathbb{N}$, $\mathcal{M}_k-$ admissible for some k. There are two possibilities, either $k \in I \setminus I'$ and then $\theta_k \displaystyle\sum_{i=1}^{n} |E_i x_2|_s \le \theta_k \sum_{i=1}^{n} |E_i x_2|'_s \le |x_2|'_{s+1}$, or $k \in I'$ and then, by hypothesis, $n = 1$, E_1 is $\mathcal{M}_k-$admissible and $\theta_k |E_1 x_2|_s \le \theta_k |x_2|_s \le |x_2|'_s \le |x_2|'_{s+1}$.

Therefore, $\|x_2\| \le \|x_2\|'$ and by $1-$unconditionality, $\|x_2\|' \le \|x\|'$. Thus, $\|x\|' \le \|x\| \le (M+1)\|x\|'$.

For (2), it is easy to see that $T(\mathcal{M}_0, \theta_0)$ is isomorphic to c_0, where $\mathcal{M}_0 = \{\{i\} \mid i \in \mathbb{N}\}$, and $\theta_0 = 1$. Now use (1) to get that $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ is isomorphic to $T\left[(\mathcal{M}_k, \theta_k)_{k\in I\cup\{0\}}\right]$ and once again to see that the latter is isomorphic to $T(\mathcal{M}_0, \theta_0)$. ∎

Proposition 2 for $I' = J$ yields

Proposition 3. *Let $J = \{k \in I \mid \theta_k = 1\}$.*

(1) Let $\displaystyle\bigcup_{k\in J} \mathcal{M}_k$ contain only a finite number of non singleton sets.
   1.1. If $J = I$, then $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ is isomorphic to c_0.
   1.2. If $J \neq I$, then $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ is isomorphic to $T\left[(\mathcal{M}_k, \theta_k)_{k\in I\setminus J}\right]$.

(2) Let $\displaystyle\bigcup_{k\in J} \mathcal{M}_k$ contain an infinite number of non singleton sets. Then $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ contains a subspace isomorphic to ℓ_1.

Proof: (1) follows from Proposition 2. For (2), we will construct a subsequence $(e_{n_i})_{i=1}^{\infty}$ of $(e_i)_{i=1}^{\infty}$ equivalent to the ℓ_1 basis.

Let $M_1 \in \bigcup_{k\in J} \mathcal{M}_k$ be a non singleton. Let $n_1 = \min M_1$. Having chosen n_i, we can take $M_{i+1} \in \bigcup_{k\in J} \mathcal{M}_k$ a non singleton such that $\min M_{i+1} > \max M_i$, and take $n_{i+1} = \min M_{i+1}$.

Consider the sequence $(e_{n_i})_{i=1}^{\infty}$ and let's show that it is equivalent to the ℓ_1 basis.

Let $x = \sum_{i=1}^{\infty} a_i e_{n_i}$. By the definition of the norm and the fact that for every $N \in \mathbb{N}$ and $i < N$, $(\{n_i\}, [n_{i+1}, n_N] \cap \mathbb{N}\})$ is $\mathcal{M}_k-$admissible for some $k \in J$ we have

$$\|x\| \geq |a_1| + \left\| \sum_{i=2}^{N} a_i e_{n_i} \right\| \geq \ldots \geq |a_1| + |a_2| + \ldots + |a_N|\,.$$

The proof is complete since always $\|x\| \leq \|x\|_1$. ∎

Observe that in statement (2) of Proposition 3 we do not ensure ℓ_1 saturation. Actually, in some cases we can also find c_0 as a subspace. This is a consequence of the following general result.

Proposition 4. *Let $\mathcal{M}_k$ be compact and hereditary for all $k \in I \subseteq \mathbb{N}$, $\theta_k \in (0,1]$ for all $k \in I$. If for all $N \in \mathbb{N}$ there exists $n \geq N$ such that for all $M \in \bigcup_{k\in I} \mathcal{M}_k$ either $n < \min M$ or $n \geq \max M$, then $T\left[(\mathcal{M}_k, \theta_k)_{k\in I}\right]$ contains a subspace isomorphic to c_0. Moreover, if $\theta_k = 1$ for all $k \in I$, the converse is true.*

Proof: We will construct a subsequence $(e_{n_i})_{i=1}^{\infty}$ of the basis $(e_i)_{i=1}^{\infty}$ equivalent to the basis of c_0.

Let $N_1 = 1$. By hypothesis there exists $n_1 \geq N_1$ such that for all $M \in \bigcup_{k\in I} \mathcal{M}_k$, $n_1 < \min M$ or $n_1 \geq \max M$.

Suppose that n_i is chosen and write $N_{i+1} = n_i + 1$. Then there exists $n_{i+1} \geq N_{i+1}$ verifying the hypothesis. Now, consider the sequence $(e_{n_i})_{i=1}^{\infty}$.

Let $x = \sum_{i=1}^{\infty} a_i e_{n_i} \in c_{00}$ and write $|x|_0 = \|x\|_{\infty}$ as in Remark 3.

Let $(E_i)_{i=1}^{n}$ be a sequence of finite subsets of $\mathbb{N}$, $\mathcal{M}_k-$ admissible for some $k \in I$. Then we have $\theta_k \sum_{i=1}^{n} |E_i x|_0 = \theta_k |E_{i_0} x|_0 \leq |x|_0$ and so $|x|_1 \leq |x|_0$. Indeed, the first equality is true since by the construction of (n_i), there exists at most one E_i such that $\mathrm{supp}(x) \cap E_i \neq \emptyset$ and the inequality is straightforward by 1-unconditionality. So we have proved that $|x|_1 = |x|_0$ and therefore $|x|_n = |x|_{n+1}$ and $\|x\| = \|x\|_{\infty}$.

The converse is a consequence of the following

CLAIM: If there is an N_0 such that for all $n \geq N_0$, there exists $M \in \bigcup_{k\in I} \mathcal{M}_k$ such that $\min M \leq n < \max M$, then every normalized block sequence in $T[(\mathcal{M}_k, 1)_{k\in I}]$ has a subsequence equivalent to the canonical basis of ℓ_1 and in particular, $T[(\mathcal{M}_k, 1)_{k\in I}]$ is ℓ_1-saturated.

Proof of CLAIM: Let $(x_i)_{i=1}^{\infty}$ be a normalized block sequence. Let i_1 be such that $N_0 \leq \min x_{i_1}$. We split $x_{i_1} = \sum_{k=p_1+1}^{p_2} a_k e_k$ in the following manner:

Let $A^{(1)}(x_{i_1}) = \left\{ j > \min x_{i_1} \mid \{t, j\} \in \bigcup_{k\in I} \mathcal{M}_k \,, t \leq \min x_{i_1} \right\}$. By hypothesis $A^{(1)}(x_{i_1})$ is not empty and $j^{(1)}(x_{i_1}) := \min A^{(1)}(x_{i_1}) > \min x_{i_1}$.

Therefore,

$$x_{i_1} = \sum_{k=p_1+1}^{p_2} a_k e_k = \sum_{k=p_1+1}^{j^{(1)}(x_{i_1})-1} a_k e_k + \sum_{k=j^{(1)}(x_{i_1})}^{p_2} a_k e_k := x_{i_1}^{(1)} + u^{(1)}.$$

Let $y_{i_1}^{(1)} = \dfrac{x_{i_1}^{(1)}}{\left\| x_{i_1}^{(1)} \right\|}$. Suppose $y_{i_1}^{(\ell)}$ is defined and we have $x_{i_1} = x_{i_1}^{(1)} + \ldots + x_{i_1}^{(\ell)} + u^{(\ell)}$. If $u^{(\ell)} \neq 0$, define $x_{i_1}^{(\ell+1)} = \left(u^{(\ell)}\right)^{(1)}$ and $y_{i_1}^{(\ell+1)} = \dfrac{x_{i_1}^{(\ell+1)}}{\left\| x_{i_1}^{(\ell+1)} \right\|}$ and keep going until we have $u^{(d_1)} = 0$ for some $d_1 \in \mathbb{N}$. Then we have $x_{i_1} = \sum_{\ell=1}^{d_1} \left\| x_{i_1}^{(\ell)} \right\| y_{i_1}^{(\ell)}$.

Now, take i_2 such that $\operatorname{supp}(x_{i_2}) > j^{(d_1)}(x_{i_1})$ and split it as before. Continuing in this manner, we obtain a sequence

$$\left(y_{i_1}^{(1)}, y_{i_1}^{(2)}, \ldots, y_{i_1}^{(d_1)}, y_{i_2}^{(1)}, \ldots, y_{i_2}^{(d_2)}, \ldots, y_{i_n}^{(1)}, \ldots, y_{i_n}^{(d_n)}, \ldots\right) := (u_k)_{k=1}^{\infty}.$$

For this sequence we have

$$\left\| \sum_{k=1}^{n} a_k u_k \right\| = |a_1| + \left\| \sum_{i=2}^{n} a_k u_k \right\| = \ldots = \sum_{k=1}^{n} |a_k|,$$

that is, $(u_k)_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_1. But $(x_{i_k})_{k=1}^{\infty}$ is a block sequence of $(u_k)_{k=1}^{\infty}$ and therefore it is also equivalent to the canonical basis of ℓ_1. ■

Remark 4.

1. *Observe that, in particular, the hypothesis of Proposition 4 implies that $i(\mathcal{M}_k) = 1$ for all $k \in I$.*
2. *The proof of the converse of Proposition 4 states that either $T[(\mathcal{M}_k, 1)_{k\in I}]$ contains a subspace isomorphic to c_0 or $T[(\mathcal{M}_k, 1)_{k\in I}]$ is ℓ_1-saturated.*

We now give an example of a Tsirelson type space which contains ℓ_1 and c_0.

Example 2. *Let $\mathcal{M} = \{F \subseteq \mathbb{N} \mid \exists i \in \mathbb{N}$ such that $F \subseteq \{2i-1, 2i\}\}$. $T(\mathcal{M}, 1)$ contains ℓ_1 by Proposition 3 and c_0 by Proposition 4. Moreover, it is easy to see that the space is isomorphic to $\ell_1 \oplus c_0$.*

2. The case $(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}$

In view of the previous results, in this section we will consider Tsirelson type spaces defined by finite sequences $(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}$, with $\theta_k \in (0,1)$ for all $k = 1, \ldots, \ell$. The main result of the section is

Theorem 1. *Let $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ and $\theta_k \in (0,1)$ for all $k = 1, \ldots, \ell$.*

1. *If $\theta_k \leq \dfrac{1}{n_k}$ for all k then $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ is c_0-saturated.*
2. *If $\theta_k > \dfrac{1}{n_k}$ for some k then $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ is ℓ_p-saturated for some $p \in (1, +\infty)$.*

Our proof of this theorem is based on Theorem 2 below, proved in [4]. In order to state it we first need some definitions.

Definition 5. *Let $m \in \mathbb{N}$ and $\phi \in K_m \setminus K_{m-1}$. An analysis of ϕ is any sequence $\{K^s(\phi)\}_{s=0}^{m}$ of subsets of K such that for every s,*

1. *$K^s(\phi)$ consists of successive elements of K_s and $\bigcup_{f\in K^s(\phi)} \operatorname{supp}(f) = \operatorname{supp}(\phi)$.*
2. *If $f \in K^{s+1}(\phi)$ then either $f \in K^s(\phi)$ or there exists k and successive $f_1, \ldots, f_d \in K^s(\phi)$ with $(\operatorname{supp}(f_1), \ldots, \operatorname{supp}(f_d))$ $\mathcal{M}_k-$admissible and $f = \theta_k(f_1 + \ldots + f_d)$.*
3. *$K^m(\phi) = \{\phi\}$.*

Definition 6.

1. *Let $\phi \in K_m \setminus K_{m-1}$ and let $\{K^s(\phi)\}_{s=0}^m$ be a fixed analysis of ϕ. Then for a given finite block sequence $(x_k)_{k=1}^{\ell}$ we set for every $k \in \{1,\ldots,\ell\}$*

$$s_k = \begin{cases} \max\{s \mid 0 \le s < m, \text{ and there are at least two } f_1, f_2 \in K^s(\phi) \\ \quad \text{such that } |\mathrm{supp}(f_i) \cap \mathrm{supp}(x_k)| > 0, i = 1,2\}, \\ \quad \text{when this set is non - empty} \\ \\ 0 \quad \text{if } |\mathrm{supp}(x_k) \cap \mathrm{supp}(\phi)| \le 1. \end{cases}$$

2. *For $k = 1,\ldots,\ell$ we define the initial part and the final part of x_k with respect to $\{K^s(\phi)\}_{s=0}^m$, and denote them respectively by x_k' and x_k'', as follows: If $\{f \in K^{s_k}(\phi) | \mathrm{supp}(f) \cap \mathrm{supp}(x_k) \neq \emptyset\} := \{f_1,\ldots,f_d\}$ with $f_1 < \ldots < f_d$, we set $x_k' = (\mathrm{supp}(f_1))x_k$ and $x_k'' = \left(\cup_{i=2}^d \mathrm{supp}(f_i)\right) x_k$.*

Notation. *Let $m \in \mathbb{N}$, $\phi \in K^m \setminus K^{m-1}$, let $\{K^s(\phi)\}_{s=0}^m$ be an analysis of ϕ, $(v_i)_{i=1}^\infty$ a block sequence and $(x_j)_{j=1}^\infty$ a block sequence with $x_j \in Span\{v_i \mid i \in \mathbb{N}\}$. Suppose that there exists n_ϕ such that $\mathrm{supp}(\phi) \subseteq \bigcup_{j=1}^{n_\phi} \mathrm{supp}(x_j)$ and denote by x_j' and x_j'' the initial and the final part of x_j, $j \le n_\phi$. For all $f = \theta_k(f_1 + \ldots + f_d) \in K^s(\phi)$ and $J \subseteq \{1,\ldots,n_\phi\}$ we define the following sets for (x_j'):*

$$I' = \{i \mid 1 \le i \le d \text{ and } \mathrm{supp}(f_i) \cap \mathrm{supp}(x_j') \neq \emptyset \text{ for at least two different } j \in J\}$$

and for every $i \in I$,

$$D_{f_i}' = \{j \in J \mid \mathrm{supp}(f_i) \cap \mathrm{supp}(x_j') \neq \emptyset \text{ and } (\mathrm{supp}(f) \cap \mathrm{supp}(x_j')) \setminus \mathrm{supp}(f_i) \subseteq \mathrm{supp}(v_t) \text{ for some } t\}$$

and

$$T' = \{j \in J \mid j \notin \bigcup_{i \in I'} D_{f_i}' \text{ and } \exists t_1 \neq t_2$$
$$\text{such that } \mathrm{supp}(x_j') \cap (\cup_{i \notin I'} \mathrm{supp}(f_i)) \cap \mathrm{supp}(v_{t_i}) \neq \emptyset, \quad i = 1,2\}.$$

In the same manner we define sets I'', D_{f_i}'', T'' exchanging x_j' for x_j''.

Theorem 2 ([4]). *Given $T[(\mathcal{M}_k,\theta_k)_{k=1}^{\ell}]$ with $\ell \in \mathbb{N}$, $\theta_k \in (0,1)$ and $i(\mathcal{M}_k) = n_k \in \mathbb{N}$, for all $k = 1,\ldots,\ell$, let $(v_i)_{i=1}^\infty$ be a normalized block sequence. If there exists a sequence $x_j = \sum_{i \in I_j} a_i v_i$ with $(a_i)_{i=1}^\infty \subset \mathbb{R}$ and $(I_j)_{j=1}^\infty \subset \mathbb{N}$ successive such that*

(a) $\dfrac{1}{2^{j+1}} \le |a_j| < \dfrac{1}{2^j}$ and

(b) for all $m \in \mathbb{N}$, $\phi \in K^m \setminus K^{m-1}$, each analysis $\{K^s(\phi)\}_{s=1}^m$ of ϕ, all $f = \theta_k(f_1 + \ldots + f_d) \in K^s(\phi)$, and all $J \subseteq \{1,\ldots,n_\phi\}$, the inequalities $|I'| + |T'| \le n_k$ and $|I''| + |T''| \le n_k$ hold,

then $(x_j)_{j=1}^\infty$ is equivalent to the canonical basis of $T[(\mathcal{A}_{n_k},\theta_k)_{k=1}^{\ell}]$.

Recall, see [4], that the space $T[(\mathcal{A}_{n_k},\theta_k)_{k=1}^{\ell}]$ is either isometrically isomorphic to c_0, when $n_k \cdot \theta_k \le 1$ for all k, or isomorphic to ℓ_p, where $p = min\left\{\dfrac{1}{1 - \log_{n_k}\frac{1}{\theta_k}} \mid n_k \cdot \theta_k > 1\right\}$. So, to prove Theorem 1 we need to find the sequence $(x_j)_{j=1}^\infty$ and the next lemma will be useful for constructing it.

Lemma 1. *Let $\ell \in \mathbb{N}$, $\theta_k \in (0,1)$ and $\mathcal{M}_k$ be such that $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ for all $k = 1,\ldots\ell$. Then for every block sequence $(u_i)_{i=1}^\infty$ in $T[(\mathcal{M}_k,\theta_k)_{k=1}^{\ell}]$ there exists an infinite subset $\mathcal{P} = \{p_i\}_{i=1}^\infty$ of $\mathbb{N}$ and a subsequence $(v_i)_{i=1}^\infty$ of $(u_i)_{i=1}^\infty$ having the following properties:*

(a) $p_1 \le \mathrm{supp}(v_1) < p_2 \le \mathrm{supp}(v_2) < \ldots < p_i \le \mathrm{supp}(v_i) < p_{i+1} \le \ldots$

(b) For every sequence $E_1 < E_2 \ldots < E_{n_k}$ of finite subsets of $\mathcal{P}$, where $E_i = \{p_{\ell_1^i},\ldots,p_{\ell_{t_i}^i}\}, i = 1,\ldots,n_k$, the family

$$\left(\bigcup_{j=\ell_1^1}^{\ell_{t_1}^1} \mathrm{supp}(v_j),\ldots,\bigcup_{j=\ell_1^{n_k}}^{\ell_{t_{n_k}}^{n_k}} \mathrm{supp}(v_j)\right)$$

is $\mathcal{M}_k-$admissible.

(c) If $r \geq n_k+1$, $S=\{s_1,\dots s_r\} \subseteq \mathbb{N}$ is such that

$$|\{j \in \mathbb{N} \mid [s_i, s_{i+1}] \cap \operatorname{supp}(v_j) \neq \emptyset\}| \geq 2$$

for all $i=1,\dots,r-1$*, then* $S \notin \mathcal{M}_k$.

Proof: The proof is based on the following result from [4]:

Lemma 2. *Let* $\ell, n_1,\dots,n_\ell \in \mathbb{N}$*. Let* $\mathcal{M}_k, k=1,\dots,\ell$ *be such that* $i(\mathcal{M}_k)=n_k$*. Then there exists an infinite subset* Q *of* $\mathbb{N}$ *having the following properties:*

1. *Let* $k \in \{1,\dots,\ell\}$*. Every sequence* $E_1 < E_2 \dots < E_{n_k}$ *of length* n_k *of finite subsets of* Q *is* $\mathcal{M}_k-$*admissible.*
2. *Let* $k \in \{1,\dots,\ell\}$*. If* $r \geq n_k+1$*, then no sequence* $E_1 < E_2 \dots < E_r$ *of finite subsets of* Q *with* $|E_i| \geq 2$ *for all* $i=1,\dots,r$*, is* $\mathcal{M}_K-$*admissible.*

Now, let $Q=\{k_i\}_{i=1}^{\infty}$ be the sequence in Lemma 2. Take $p_1=k_1$, and $v_1=u_\ell$ such that $p_1 \leq \operatorname{supp}(u_\ell)$. Having chosen p_i and v_i with $p_i \leq \operatorname{supp}(v_i)$, since $\{k_i\}_{i=1}^{\infty}$ is increasing, let k_{j_i} be such that $p_i \leq \operatorname{supp}(v_i) < k_{j_i}$, and take $p_{i+1}=k_{j_i+1}$ and $v_{i+1}=u_\ell$ such that $p_{i+1} \leq \operatorname{supp}(u_\ell)$.

The sequences $\{p_i\}_{i=1}^{\infty}$ and $(v_i)_{i=1}^{\infty}$ satisfy the assertions of Lemma 1:

(a) By construction.

(b) It is sufficient to see that $\bigcup_{j=\ell_1^i}^{\ell_{t_i}^i} \operatorname{supp}(v_j) \subseteq \left[p_{\ell_1^i}, p_{\ell_{t_i}^i}\right]$ and, since the family $\left\{\left\{p_{\ell_1^i}, p_{\ell_{t_i}^i}\right\}\right\}_{i=1}^{n_k}$ is $\mathcal{M}_k-$admissible by Lemma 2, $\left(\bigcup_{j=\ell_1^i}^{\ell_{t_i}^i} \operatorname{supp}(v_j)\right)_{i=1}^{n_k}$ is also $\mathcal{M}_k-$admissible.

(c) Let $S=\{s_1,\dots,s_r\}$ be such that $|\{j \in \mathbb{N} \mid [s_i,s_{i+1}] \cap \operatorname{supp}(v_j) \neq \emptyset\}| \geq 2$ for all $i=1,\dots,r-1$, let $d_i=\min\{j \in \mathbb{N} \mid [s_i,s_{i+1}] \cap \operatorname{supp}(v_j) \neq \emptyset\}$. Then $k_{j_{d_i}}$ and $p_{d_i+1} \in [s_i,s_{i+1}] \cap Q$ for all $i=1,\dots,r-1$, and by the property (2) of Lemma 2, $S \notin \mathcal{M}_k$. ∎

Proof: (of Theorem 1). It suffices to show that c_0 or ℓ_p is included in every block subspace of $T[(\mathcal{M}_k,\theta_k)_{k=1}^{\ell}]$.

Let $(u_i)_{i=1}^{\infty}$ be a normalized block sequence. Let $\mathcal{P}=\{p_i\}_{i=1}^{\infty}$ and $(v_i)_{i=1}^{\infty}$ be the sequences associated to $(u_i)_{i=1}^{\infty}$ from Lemma 1.

If $\sup_{m\in\mathbb{N}} \left\|\sum_{i=1}^{m} v_i\right\|$ is finite, then $(v_i)_{i=1}^{\infty}$ is equivalent to the canonical basis of c_0, and from Corollary 1 of [4] we have $n_k \cdot \theta_k \leq 1$.

Suppose now that $\lim_{m\to\infty} \left\|\sum_{i=1}^{m} v_i\right\| = \infty$. Then we can construct a sequence $(y_i)_{i=1}^{\infty}$ supported by the subsequence $(v_i)_{i=1}^{\infty}$ with the following properties: For every $j, y_j = \frac{1}{2^{j+1}} \sum_{i\in I_j} v_i$, where

(i) I_j are successive intervals of $\mathbb{N}$, and

(ii) $1-\frac{1}{2^{j+1}} \leq \|y_j\| \leq 1$.

If $x_j = \frac{y_j}{\|y_j\|}$, the sequence x_j satisfies condition (a) of Theorem 2.

We prove condition (b) of Theorem 2 for the initial parts of (x_j) since for the final parts the proof is analogous. Suppose that ϕ, f and J are fixed. Let $m_1 \leq \operatorname{supp}(f_1) < m_2 \leq \operatorname{supp}(f_2) < \ldots < m_d \leq \operatorname{supp}(f_d)$. We define $B \subseteq \{m_1,\dots,m_d\}$ as follows:

$$m_{i_s} \in B \Longleftrightarrow \begin{cases} i_s \in I', \\ i_s = \min\{i \notin I' \mid \operatorname{supp}(x'_j) \cap \operatorname{supp}(f_i) \neq \emptyset\} \text{ for some } j \in T'. \end{cases}$$

Let $m_{i_1} < \ldots < m_{i_r}$ be the elements of B. Observing that

$$\left|\left\{t \in \mathbb{N} \mid [m_{i_s}, m_{i_{s+1}}] \cap \mathrm{supp}(v_t)\right\}\right| \geq 2, \qquad \forall\ 1 \leq s \leq r-1$$

and using property (c) of Lemma 1 we get that $r = |B| \leq n_k$. So $|I'| + |T'| \leq n_k$. ∎

The proof of the next two corollaries easily follows from Theorem 1 from this paper and Corollaries 1 and 2 from [4].

Corollary 1. *Let $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$, $1 < p < \infty$, $n_k = i(\mathcal{M}_k)$ and $\theta_k \in (0,1)$. The following conditions are equivalent:*

i) $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ contains a subspace isomorphic to ℓ_p.

ii) $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ is ℓ_p-saturated.

iii) $i(\mathcal{M}_k)$ is finite, $\theta_k > \dfrac{1}{n_k}$ for some $k = 1, \ldots, \ell$ and $p = \min\left\{\dfrac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \mid n_k \cdot \theta_k > 1\right\}$.

Corollary 2. *Let $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$, $\theta_k \in (0,1)$. The following conditions are equivalent:*

i) $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ contains a subspace isomorphic to c_0.

ii) $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ is c_0-saturated.

iii) $i(\mathcal{M}_k)$ is finite and $\theta_k \leq \dfrac{1}{i(\mathcal{M}_k)}$ for all $k = 1, \ldots, \ell$.

In view of Proposition 1 and the previous corollaries we can include the case ℓ_1 in the discussion.

Corollary 3. *Let $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$, $2 \leq i(\mathcal{M}_k) \in \mathbb{N}$ and $\theta_k \in (0,1]$. The following conditions are equivalent:*

i) $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ contains a subspace isomorphic to ℓ_1.

ii) $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ is ℓ_1-saturated.

iii) $\theta_k = 1$ for some $k = 1, \ldots, \ell$

So in particular we have proved the following criterion, which is useful to show when two Tsirelson type Banach spaces are totally incomparable.

Theorem 3. *Let $\ell, \ell' \in \mathbb{N}$, $\theta_k \in (0,1)$ and $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \ldots, \ell$ and $\theta'_k \in (0,1)$ and $i(\mathcal{M}'_k) = n'_k \in \mathbb{N}$ for all $k = 1, \ldots, \ell'$. Then $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ and $T[(\mathcal{M}'_k, \theta'_k)_{k=1}^{\ell'}]$ are totally incomparable if and only if one of the following situations occurs:*

1. *$\theta_k \leq \dfrac{1}{n_k}$ for all $k = 1, \ldots, \ell$ and $\theta'_k > \dfrac{1}{n'_k}$ for some $k \in \{1, \ldots, \ell'\}$, or*
2. *$\theta'_k \leq \dfrac{1}{n'_k}$ for all $k = 1, \ldots, \ell'$ and $\theta_k > \dfrac{1}{n_k}$ for some $k \in \{1, \ldots, \ell\}$, or*
3. *$\theta_k > \dfrac{1}{n_k}$ for some $k \in \{1, \ldots, \ell\}$ and $\theta'_k > \dfrac{1}{n'_k}$ for some $k \in \{1, \ldots, \ell'\}$ and*

$$\min\left\{\frac{1}{1 - \log_{n_k} \frac{1}{\theta_k}} \mid n_k \cdot \theta_k > 1\right\} \neq \min\left\{\frac{1}{1 - \log_{n'_k} \frac{1}{\theta'_k}} \mid n'_k \cdot \theta'_k > 1\right\}.$$

Also we obtain a characterization of the reflexivity of this kind of spaces as in [1].

Proposition 5. *Let $\ell \in \mathbb{N}$. Let $\theta_k \in (0,1)$ and $i(\mathcal{M}_k) = n_k \in \mathbb{N}$ for all $k = 1, \ldots \ell$. Then the following conditions are equivalent:*

1. *$T[(\mathcal{M}_k, \theta_k)_{k=1}^{\ell}]$ is reflexive.*
2. *$\theta_k > \dfrac{1}{i(\mathcal{M}_k)}$ for some $k \in \{1, \ldots, \ell\}$.*

3. A criterion of total incomparability for spaces of the form $T[(\mathcal{A}_k, \theta_k)_{k=1}^{\infty}]$

We will suppose throughout the section that $(\theta_k)_{k=1}^{\infty} \subset (0,1]$ is a non increasing null sequence since $T[(\mathcal{A}_k, \theta_k)_{k=1}^{\infty}]$ is easily seen to be isometric to $T[(\mathcal{A}_k, \theta'_k)_{k=1}^{\infty}]$ where $\theta'_k = \sup\{\theta_j \mid j \geq k\}$ and $\inf\{\theta_k\} > 0$ implies that $T[(\mathcal{A}_k, \theta_k)_{k=1}^{\infty}]$ is isomorphic to ℓ_1.

The following properties of such spaces, stated as lemmas, are known.

Lemma 3. *Let $(u_i)_{i=1}^n$ be a normalized block sequence in $T[(\mathcal{A}_k,\theta_k)_{k=1}^\infty]$. Then for all $a_1,\ldots,a_n \in \mathbb{R}$,*

$$\left\|\sum_{i=1}^n a_i e_i\right\| \le \left\|\sum_{i=i}^n a_i u_i\right\|.$$

Proof: It is easy to prove by induction on s that $\left|\sum_{i=1}^n a_i e_i\right|_s \le \left\|\sum_{i=i}^n a_i u_i\right\|$. ∎

The following lemma was proved in [11] with $\theta_k = (\log_2(1+k))^{-1}$, but the same proof works for any θ_k converging to zero.

Lemma 4 ([11]). *Let $T\left[(\mathcal{A}_k,\theta_k)_{k=1}^\infty\right]$, let θ_k converge to 0. Let $(y_n)_{n=1}^\infty$ be a block sequence, let a strictly decreasing null sequence $(\varepsilon_n)_{n=1}^\infty \subset \mathbb{R}^+$ and a strictly increasing sequence $(k_n)_{n=1}^\infty \subset \mathbb{N}$ be such that for each n there is a normalized block sequence $(y(n,i))_{i=1}^{k_n}$, $(1+\varepsilon_n)-$equivalent to the $\ell_1^{k_n}$ basis and $y_n = \dfrac{1}{k_n}\sum_{i=1}^{k_n} y(n,i)$. Then for all $\ell \in \mathbb{N}$,*

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\ldots\lim_{n_\ell\to\infty}\left\|\sum_{i=1}^\ell y_{n_i}\right\| = \left\|\sum_{i=1}^\ell e_i\right\|.$$

We will consider spaces such that ℓ_1 is finitely block represented in every block subspace of the space but not containing ℓ_1. The role of ℓ_1 in this context, as well as that of c_0, can be easily described:

Proposition 6. *The following conditions are equivalent:*
i) The identity is an isometric isomorphism from $T[(\mathcal{A}_k,\theta_k)_{k=1}^\infty]$ onto c_0.
ii) $T[(\mathcal{A}_k,\theta_k)_{k=1}^\infty]$ contains a subspace isomorphic to c_0.
iii) For all $n\in\mathbb{N}$, $\|\sum_{i=1}^n e_i\| = 1$.
iv) $\theta_k \le 1/k$ for all $k\in\mathbb{N}$.

Proof: *ii)* $\Rightarrow$ *iii)*: By the Bessaga-Pelcynski Principle and a theorem of R.C. James (see e.g. [8], pg. 97), for every $\varepsilon > 0$ there exists a normalized block sequence $(u_i)_{i=1}^\infty$ such that for all $\ell\in\mathbb{N}$,

$$\max|a_i| \le \|\sum_{i=1}^\ell a_i u_i\| \le (1+\varepsilon)\max|a_i| \qquad a_1\ldots a_\ell \in \mathbb{R}$$

and so by Lemma 3, $\|\sum_{i=1}^\ell e_i\| \le (1+\varepsilon)$ and iii) follows. *iii)* $\Rightarrow$ *iv)*: This is clear since $\theta\cdot\ell \le \|\sum_{i=1}^\ell e_i\|$. *iv)* $\Rightarrow$ *i)*: By induction on $m\in\mathbb{N}$ it easily follows that $|\cdot|_m = |\cdot|_0$ on c_{00}. ∎

Proposition 7. *Let $T\left[(\mathcal{A}_k,\theta_k)_{k=1}^\infty\right]$, let θ_k converge to 0 . The following conditions are equivalent:*
i) The identity is an isometric isomorphism from $T\left[(\mathcal{A}_k,\theta_k)_{k=1}^\infty\right]$ onto ℓ_1.
ii) $T\left[(\mathcal{A}_k,\theta_k)_{k=1}^\infty\right]$ contains a subspace isomorphic to ℓ_1.
iii) For all $n\in\mathbb{N}$, $\|\sum_{i=1}^n e_i\| = n$.
iv) $\theta_2 = 1$.

Proof: *ii)* $\Rightarrow$ *iii)*. Choose a strictly decreasing sequence $(\varepsilon_n))_{n=1}^\infty \subset \mathbb{R}_+$ converging to 0 and $k_n = n$. We will construct a block sequence $(y_n)_{n=1}^\infty$ as in Lemma 4 above.

By James' Theorem let $(u_i))_{i=1}^\infty$ be a normalized block sequence $(1+\varepsilon_1)-$equivalent to the unit vector basis of ℓ_1. Let $y_1 = u_1$. Again by James' theorem there exist a normalized block sequence $(u_i'))_{i=1}^\infty$ with $u_i' \in Span\{u_i \mid i\in\mathbb{N}\}$ and $y_1 < u_1'$, $(1+\varepsilon_2)-$equivalent to the unit vector basis of ℓ_1. Let $y_2 = \frac{1}{2}(u_1'+u_2')$. We continue in the same way.

Let $\ell\in\mathbb{N}$. Since any block sequence $(y_{n_i})_{i=1}^\ell$ is $(1+\varepsilon_1)-$equivalent to the unit vector basis of ℓ_1^ℓ, by Lemma 4 we have

$$(1-\varepsilon_1)\ell \le \|\sum_{i=1}^\ell e_i\| \le \ell$$

and the result follows. $iii) \Rightarrow iv)$: Just notice that $2 = \|e_1 + e_2\| = 2\theta_2$. $iv) \Rightarrow i)$: This follows by induction on $|\operatorname{supp}(x)|$. ∎

We now give sufficient conditions, in terms of the behaviour of $\lambda_n := \left\|\sum_{i=1}^{n} e_i\right\|$, guaranteeing that in a space of this kind ℓ_1 is finitely block represented in every block subspace.

Proposition 8 ([5]). *Let $n, \ell \in \mathbb{N}, 0 < \varepsilon < 1$. Let $(X, \|\cdot\|)$ be a normed space with a normalized 1-unconditional normalized basis $(e_i)_{i=1}^{n^\ell}$ such that*

$$(n-\varepsilon)^\ell \le \left\|\sum_{i=1}^{n^\ell} e_i\right\| \le n^\ell.$$

Then there exists a normalized block sequence $(y_i)_{i=1}^n$ of $(e_i)_{i=1}^{n^\ell}$ such that

$$n - \varepsilon \le \left\|\sum_{i=1}^{n} y_i\right\| \le n.$$

Moreover, $(y_i)_{i=1}^n$ is $\frac{1}{1-\varepsilon}$-equivalent to the canonical basis of ℓ_1^n.

Proposition 9. *Let $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$, let θ_k converge to 0. If there exists $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ unbounded and $(\ell_k)_{k=1}^\infty$ such that*

$$\lim_{k\to\infty}\left[n_k - \left(\lambda_{n_k^{\ell_k}}\right)^{\frac{1}{\ell_k}}\right] = 0,$$

then ℓ_1 is finitely block represented in every block subspace of $T[(\mathcal{A}_k, \theta_k)_{k=1}^\infty]$.

Proof: Given $n \in \mathbb{N}$ and $0 < \varepsilon < 1$, take $k \in \mathbb{N}$ such that $n_k > n$ and $n_k - \left(\lambda_{n_k^{\ell_k}}\right)^{\frac{1}{\ell_k}} < \varepsilon$. Let $(u_i)_{i=1}^\infty$ be a normalized block sequence. Then

$$n_k^{\ell_k} \ge \left\|\sum_{i=1}^{n_k^{\ell_k}} u_i\right\| \ge \left\|\sum_{i=1}^{n_k^{\ell_k}} e_i\right\| = \lambda_{n_k^{\ell_k}} \ge (n_k - \varepsilon)^{\ell_k}$$

and, by Proposition 8, $\ell_1^{n_k}$ is finitely block represented in blocks of $(u_i)_{i=1}^\infty$. ∎

Remark 5. *By similar arguments it is easy to prove that the following condition is also sufficient:*

1. *There exits $m \ge 2$ such that $\lim_{\ell\to\infty} (\lambda_{m^\ell})^{\frac{1}{\ell}} = m$.*

We can also give sufficient conditions for the sequence $(\theta_k)_{k=1}^\infty$:

2. There exists $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ unbounded and $(\ell_k)_{k=1}^\infty$ such that $\lim_{k\to\infty} n_k \left[1 - \left(\theta_{n_k^{\ell_k}}\right)^{\frac{1}{\ell_k}}\right] = 0$ or
3. There exists $m \ge 2$ such that $\lim_{\ell\to\infty} (\theta_{m^\ell})^{\frac{1}{\ell}} = 1$ or, equivalently, $\lim_{\ell\to\infty} (\theta_{m^\ell})^{\frac{1}{\ell}} = 1$ for all $m \ge 2$.

Lemma 5. *Let $(X, \|\cdot\|)$ and $(X', \|\cdot\|')$ be Banach spaces not totally incomparable with Schauder bases $(e_i)_{i=1}^\infty$ and $(e'_i)_{i=1}^\infty$. If $(e_i)_{i=1}^\infty$ is shrinking, there exist block sequences $(u_i)_{i=1}^\infty$ and $(u'_i)_{i=1}^\infty$ of $(e_i)_{i=1}^\infty$ and $(e'_i)_{i=1}^\infty$ respectively such that the application $T : \overline{Span}\{u_i \mid i \in \mathbb{N}\} \longrightarrow \overline{Span}\{u'_i \mid i \in \mathbb{N}\}$, given by $T(u_i) = u'_i$ for all $i \in \mathbb{N}$ is an isomorphism.*

Proof: There exist subspaces $Y \subseteq X$ and $Y' \subseteq X'$ and an isomorphism $S : Y \longrightarrow Y'$. We will see that for all $\varepsilon > 0$ we can find block sequences $(u_i)_{i=1}^\infty$ and $(u'_i)_{i=1}^\infty$ such that $(1-\varepsilon)\|S\|\|S^{-1}\| \le \|T\|\|T^{-1}\| \le (1+\varepsilon)\|S\|\|S^{-1}\|$.

Let $\varepsilon > 0$. There exists a normalized block sequence $(x_i)_{i=1}^\infty$ of $(e_i)_{i=1}^\infty$ and $\overline{Span}\{y_i \mid i \in \mathbb{N}\} \subseteq Y$ such that the linear isomorphism defined by $U(x_i) = y_i$ verifies $\|U\|\|U^{-1}\| \le 1 + \varepsilon$. Let $y'_i := S(y_i)$ for all $i \in \mathbb{N}$.

Since $\inf_{i\in\mathbb{N}} \|y'_i\| > 0$ and $(e_i)_{i=1}^\infty$ is a shrinking basis, y'_i tends to 0 weakly. So, by the Bessaga-Pelcynski principle, there is a subsequence $(y'_{i_k})_{k=1}^\infty$ and a block sequence $(u'_k)_{k=1}^\infty$ of $(e'_i)_{i=1}^\infty$ such that the isomorphism defined by $V(y'_{i_k}) = u'_k$ verifies $\|V\|\|V^{-1}\| \le 1 + \varepsilon$. Take $u_k = x_{i_k}$ and $T = V \circ S \circ U$. ∎

Remark 6. *Let $X = T[(\mathcal{A}_k, \theta_k)_{k=1}^{\infty}], \theta_k \in (0,1)$. Since its canonical basis $(e_i)_{i=1}^{\infty}$ is unconditional, hence being shrinking is equivalent to ℓ_1 not being isomorphic to any subspace of X and this is the case by Proposition 7.*

Theorem 4. *Let $X = T[(\mathcal{A}_k, \theta_k)_{k=1}^{\infty}]$ and $X' = T[(\mathcal{A}_k, \theta'_k)_{k=1}^{\infty}]$ with $\theta_k, \theta_{k'} \in (0,1)$ be such that ℓ_1 is finitely block represented in every block subspace of X and X'. If X and X' are not totally incomparable, then there exists $C \geq 0$ such that for all $n \in \mathbb{N}$,*

$$(*) \qquad \frac{1}{C} \leq \frac{\lambda_\ell}{\lambda'_\ell} \leq C.$$

Proof: Denote by $\|\cdot\|$ and $\|\cdot\|'$ the norms of X and X', respectively. By Lemma 5, there exist block sequences $(u_i)_{i=1}^{\infty} \subseteq X$ and $(u'_i)_{i=1}^{\infty} \subseteq X'$ of their respective bases denoted by $(e_i)_{i=1}^{\infty}$ and $(e'_i)_{i=1}^{\infty}$, such that $T : \overline{Span}\{u_i \mid i \in \mathbb{N}\} \longrightarrow \overline{Span}\{u'_i \mid i \in \mathbb{N}\}$, given by $T(u_i) = u'_i$ for all $i \in \mathbb{N}$ is an isomorphism. Therefore, for all $(a_i)_{i=1}^{\infty} \subseteq \mathbb{R}$ and $n \in \mathbb{N}$,

$$\frac{1}{\|T\|} \left\| \sum_{i=1}^{n} a_i u'_i \right\|' \leq \left\| \sum_{i=1}^{n} a_i u_i \right\| \leq \|T^{-1}\| \ \left\| \sum_{i=1}^{n} a_i u'_i \right\|'.$$

By Lemma 4, given $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exists a normalized block sequence $y_1, \dots, y_\ell$ of $(u_i)_{i=1}^{\infty}$, such that

$$\lambda_\ell - \varepsilon \leq \left\| \sum_{i=1}^{\ell} y_i \right\| \leq \lambda_\ell + \varepsilon.$$

Let $y'_i := T(y_i)$ for all $i = 1, \dots, \ell$. Then we have

$$\begin{aligned}
\lambda_\ell + \varepsilon &\geq \left\| \sum_{i=1}^{\ell} y_i \right\| \geq \frac{1}{\|T\|} \left\| \sum_{i=1}^{\ell} y'_i \right\|' = \\
&= \frac{1}{\|T\|} \left\| \sum_{i=1}^{\ell} \|y'_i\|' \frac{y'_i}{\|y'_i\|'} \right\|' \geq \frac{1}{\|T\|} \min_{1 \leq i \leq \ell} \|y'_i\|' \left\| \sum_{i=1}^{\ell} \frac{y'_i}{\|y'_i\|'} \right\|' \geq \\
&\geq \frac{1}{\|T\| \|T^{-1}\|} \left\| \sum_{i=1}^{\ell} e'_i \right\|' = \frac{1}{\|T\| \|T^{-1}\|} \lambda'_\ell
\end{aligned}$$

(note that in the last inequality we use Lemma 3). Since the inequality is true for all $\varepsilon > 0$, we have proved that $\lambda_\ell \geq \dfrac{1}{\|T\| \|T^{-1}\|} \lambda'_\ell$.

Now we reverse the roles of X and X' to obtain $\dfrac{1}{\|T\| \|T^{-1}\|} \lambda'_\ell \leq \lambda_\ell \leq \|T\| \|T^{-1}\| \lambda'_\ell$. ∎

Remark 7. *If X and X' contain isometric subspaces Y and Y' , then $\lambda_\ell = \lambda'_\ell$ for all $\ell \in \mathbb{N}$. Actually, the same equality holds if for every $\varepsilon > 0$, X and X' contain $(1+\varepsilon)-$isomorphic subspaces.*

Remark 8. *There are special cases when the calculus of λ_ℓ is easy. For instance when $(\theta_k), (\theta'_k)$ belong to the so called class $\mathcal{F}$ defined in [11] we have $\lambda_\ell = \ell \cdot \theta_\ell$ and the condition $(*)$ of Theorem 4 yields $\dfrac{1}{C} \leq \dfrac{\theta_\ell}{\theta'_\ell} \leq C$ for all ℓ or $\theta_\ell = \theta'_\ell$ if we can find isometric subspaces or $(1+\varepsilon)-$isomorphic subspaces for all $\varepsilon > 0$.*

Example 3. *Let $f_r(x) = \log_2^r(1+x)$ with $0 < r < 3\log 2 - 1$. Then $(f_r^{-1}(k)) \in \mathcal{F}$ and if $0 < r < s < 3\log 2 - 1$, the spaces $T\left[\left(\mathcal{A}_k, \dfrac{1}{f_r(k)}\right)_{k=1}^{\infty}\right]$ and $T\left[\left(\mathcal{A}_k, \dfrac{1}{f_s(k)}\right)_{k=1}^{\infty}\right]$ are, by Theorem 4, totally incomparable. Moreover, it is easy to check that these spaces are also totally incomparable to ℓ_p, $1 \leq p < \infty$ or c_0.*

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